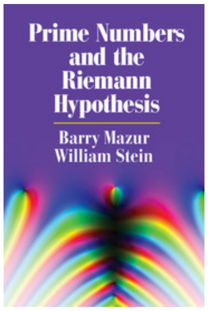
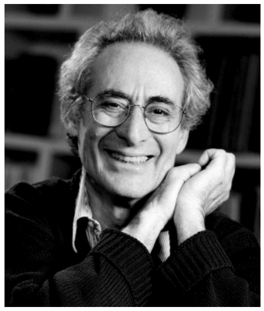
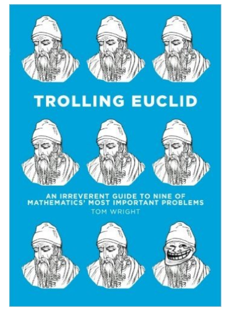
Prime numbers and the Riemann hypothesis, *Barry Mazur & William Stein*, Cambridge University Press, (2016) ISBN 978-1107499430 (pbk), xii+142 pp
Trolling Euclid, *Tom Wright*, CreateSpace Independent Publishing Platform (2016) ISBN 978-1523466467 (pbk), 206 pp.



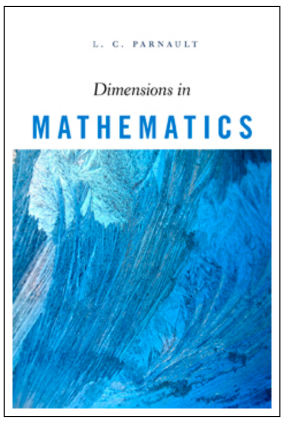



B. Mazur & W. Stein



Tom Wright

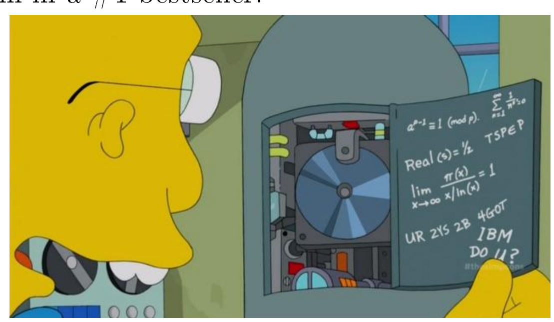
A popular book about mathematics is most often dealing with prime numbers or has at least a chapter devoted to it. The choice is obvious because anybody knows what the positive integers are and almost everybody has an idea of what prime numbers are. And that is usually where it stops for the average (wo)man in the street. At best, they have heard about the proof of Fermat's Last Theorem by Andrew Wiles in 1994. The possibility you find somebody who knows or can formulate the Riemann Hypothesis is extremely small, unless you are asking people at the exit of a mathematics building at lunch time.



April's fool blog 2013 of Harvard U. Press

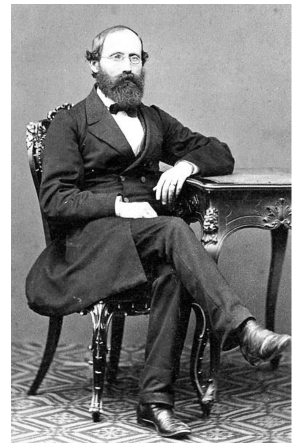
Millions have read Stieg Larsson's *Millennium* trilogy or seen the movie(s), but they probably did not quite understand why in volume 2 Lisbeth Salander starts thinking about FLT after reading *THE* book about mathematics, *Dimensions in Mathematics*, a 1200 page mathematical bible by L.C. Parnault, pleasant to read and amply illustrated, where you read about *'Archimedes, Newton, and Martin Gardner, and dozens of other classical mathematicians'*. (Un?)fortunately such book only exists in fiction. At some point Salander even has the same insight as Fermat had, when he wrote that he *discovered a truly marvelous proof of this, which this margin is too narrow to contain.* She suddenly realizes that '*The answer was so disarmingly simple.* [...] *No wonder mathematicians were tearing out their hair.*' She however got shot in the head, and later could not immediately recall her solution. She lost interest anyway since she had solved it at some point, and then there was no more motivation to re-solve the riddle. How trendy can mathematics be if it can make it as a nonsense item in a #1 bestseller.

But back to the RH. Asking around, you might find some people who know that the distribution of prime numbers has some strange regularities, yet behaves totally unpredictable, somewhat like the digits of π. Formulating the RH would still be a problem, in particular since its usual formulation does not look like it has anything to do with prime numbers. Suppose your interviewee were interested to learn about it, then the booklet by Mazur and Stein is precisely what you should recommend. The RH is not in Larsson's Millennium trilogy, but it is one of the *Millennium Prize Problems* of the *Clay Mathematical Institute* in 2000, a century after David Hilbert had listed it among the most important mathematical problems in 1900. Trying to solve it is still one of the most difficult ways to earn yourself a million dollars.



In *The Simpsons and their Mathematical secrets* S. Singh discusses RH in Simpsons-Futurama

There are several ways to introduce the RH. In most cases one starts from the summation $\sum_{k=1}^{\infty} 1/k^s$ to define it as the function $\zeta(s)$, after extending this to complex s values, everywhere in $\mathbb{C}$ except $s = 1$ (perhaps introducing the surprising fact that $\sum_{k=1}^{\infty} k = -1/12$) and finally arrive at the problem about proving the location of its nontrivial zeros on the axis $\mathrm{Re}\, s = 1/2$ in the complex plane. In this approach, it comes as a surprise that this has anything to do with prime number distribution. Then one needs to introduce the marvelous Euler formula $\zeta(s) = \prod_{p \text{ prime}} 1/(1-p^{-s})$. This is more or less the approach taken by E. Frenkel in his Numberphile video blog[1].



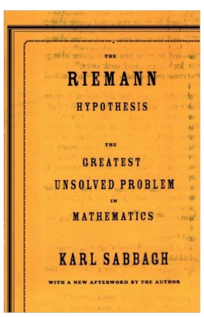
Bernard Riemann

This is not the approach taken by the authors of this marvelous booklet. They start from prime numbers and stick to this idea till the end. The book is written for a broad audience, but it has some parts that require more mathematics. That is why they have subdivided their text in four parts. The first part is intended for the non-mathematician. It takes about half of the book and goes all the way from the history and importance of the RH and prime numbers, to the staircase function $\pi(x)$ counting all primes less than x, its square root approximations, namely Gauss' $x/(\log x - 1)$ and Riemann's logarithmic integral $Li(x)$. Then $\pi(x)$ needs a modification to include powers of primes and the use of logarithmic scales to obtain a function $\psi(x)$ which looks approximately like a straight line at a 45 degree angle. Eventually Fourier analysis is used to hint that the spectrum of a related distribution will reveal the distribution of the prime numbers. That is where the reader of part I is left, with Fourier as teaser to read on.

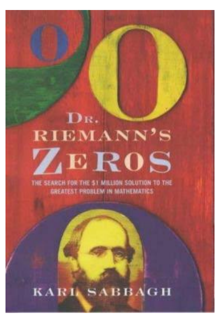
But the continuation requires more mathematics. So part II is preparatory, introducing generalized functions or distributions and their Fourier transforms. Some manipulation of the $\psi(x)$ will give a function Ψ whose derivative gives spikes at the positions of the logarithm of prime numbers and their integer multiples. The details are less easy to follow, but it is clear that its spectrum defines the location of the primes and their powers. Riemann's approach via the zeta function is only introduced in the trailing chapters of part IV. It then takes the approach of Frenkel as sketched above to come to the link between the nontrivial zeros of the zeta function and the distribution of the primes.

It is a nice, amply illustrated, little booklet that contains surprisingly much information brought at a level accessible for many kinds of readers. The mathematics are somewhat smuggled under the carpet but there are many graphs that should somehow convince the reader. It may become a bit fuzzy near the end for readers not well prepared. It does illustrate the importance of the RH since many very different yet equivalent theorems exist and many other theorems start with 'Assume that the RH is true, then...'. And of course, there is still one million dollars waiting for you if you are interested.

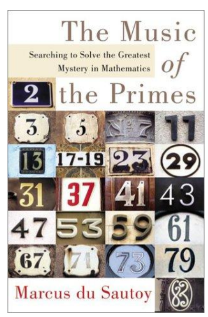
Some other popular books on the Riemann Hypothesis:



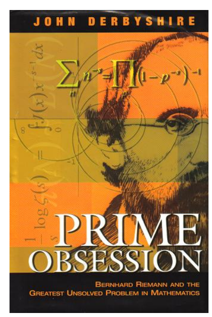
K. Sabbagh (2003)



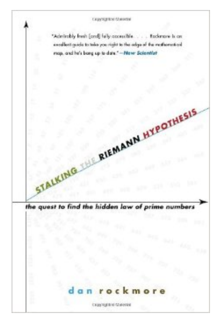
K. Sabbagh (2003)



M. du Sautoy (2003)



J. Derbyshire (2003)



D. Rockmore(2005)

The second book, *Trolling Euclid* is a bit similar because it is an airy collection of short chapters introducing the reader to a number of open problems in mathematics. Tom Wright is a number theorist at the Wofford University in Spartanburg, SC. The reason for his book sounds familiar to mathematicians: when people ask him about his job and he says he's a mathematician, he gets some frowns, and when he confirms that he does number theory, not immediately recognized as applied mathematics that is useful for anything practical, he has to explain. So he wrote this book, not to be preachy or teaching the mathematics. Instead he is entertaining, telling his thing in a conversation-like way, and with a lot of humor and self-reflection, like small-talking during a reception. So it is more entertaining and less convincing than the previously reviewed book, but on the other hand it touches on more types of mathematical problems.

The Riemann Hypothesis and its generalization are the first two problems considered. Here the start is directly from the zeta function. Chapter titles like "*The zeta function: Magical, mystical, and... dear god, what is this thing?*" or "*Wait, wait, that's it? The question of when some esoteric function hits zero is*

[1] https://www.youtube.com/watch?v=d6c6uIyieoo

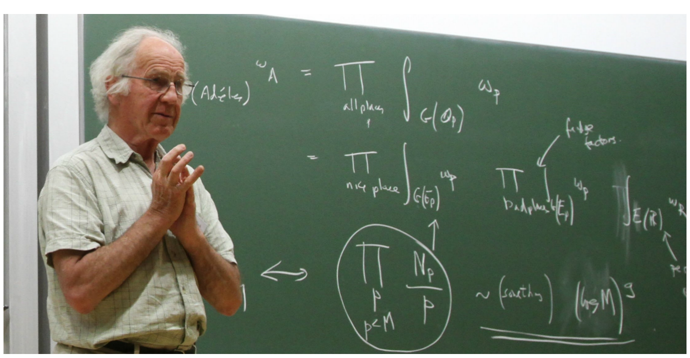
the most important problem in math?" set the tone of the book. More examples to follow. Also here some more mathematical parts, like for example analytic continuation, are extra chapters labelled 'Appendix' that can be skipped. The connection to prime numbers is seen as an application. This link is restricted to the formulation of the fact that $\pi(x)$ and $Li(x)$ will never differ by more than about $\sqrt{x}\ln x$ if and only if the RH holds.

The second problem is the GENERALIZED RH ("*How much harder can we make this stupid thing, anyway?*"). What if we replace the numerators 1 in $\sum_k(1/k^s)$ by some pattern like a sequence of alternating 0 and 1, or a repetition of the pattern $\chi_5 = (1, i, -i, -1, 0)$ and consider one of Dirichlet's L-functions $L(s,\chi_n)$, with χ_n periodic of length n? The GRH is formulated by Adolf Piltz in 1884. Wright claims that "*Piltz, as you no doubt recall, was not the most adroit when it came to manipulation of these functions, so he did the next best thing; he grabbed $L(s,\chi_n)$, put it in chokehold, and said 'TELL ME WHERE YOUR ZEROS ARE!'. Unfortunately Piltz was a bit too strong for his own good, and $L(s,\chi_n)$ was only able to respond 'Mmfghh wmmph thffff...' before passing out*". Anyway the GRH says that these functions have properties very similar to $\zeta(s)$ with nonnegative zeros all on the same vertical axis at $\operatorname{Re} s = 1/2$. If true, it gives extra information about the prime number distribution. Consider a fixed number m and denote by $\pi(x,m,n)$ the number of primes less than x of the form $n(\text{mod } m)$, then $\pi(x,m,n_1)$ and $\pi(x,m,n_2)$ do not differ by more than $\sqrt{x}$ which generalized the Prime Number Theorem. And there are a number of other consequences that are also discussed like the maximal gap between prime numbers.

Shinichi Mochizuki

The next open problem is the ABC CONJECTURE ("*What the alphabet looks like when D through Z are eliminated*"). This is relatively recent (formulated in 1985 by Joseph Oesterlé and later by David Masser). Denote $\text{rad}(n)$ for the product of all the *different* primes that divide n. If three coprime numbers satisfy $a+b=c$ then for all $\epsilon > 0$ there are only finitely many triples such that $c > \text{rad}(abc)^{1+\epsilon}$. In 2012 Shinichi Mochizuki announced a proof using a totally original approach called inter-universal Teichmüller theory (IUT). An error was detected in his proof, but nobody was familiar with IUT, since it was a private Mochizuki invention, it will take a while to verify or possibly complete his proof. Again some consequences of the ABC conjecture are listed among which FLT. Unfortunately it doesn't hold the other way around. In an appendix chapter, it is shown that deriving FLT is an easy consequence since $x^n + y^n = z^n$ is indeed of the form $a+b=c$. In fact ABC-type claims hold for many other equations of the form $a+b=c$ outside number theory.

The BIRCH-SWINNERTON-DYER CONJECTURE is another of the Millennium Problems, formulated in the 1960's. Wright gives the following loose introduction. Consider an elliptic curve E of the form $y^2 = x^3 + Ax^2 + Bx + C$ with A, B, C integers. The problem is to know whether there are infinitely many rational points on E. Gauss proved that if there is no solution modulo n, then there is no solution at all. But what if there are some? Let N_p be the number of solutions modulo a prime number p. These numbers are smuggled into a formula of the type of the Dirichlet L-functions. Let's call this $L_E(s)$. It is defined for every s, even $s = 1$ ("*Put that in your pipe and smoke it, Riemann*"). In fact BSD says that $L_E(1) = 0$, if and only if E has infinitely many rational points. One direction is proved in the Coates-Wiles[2] theorem which says that if $L_E(1) \neq 0$ then there are not infinitely many rational points.



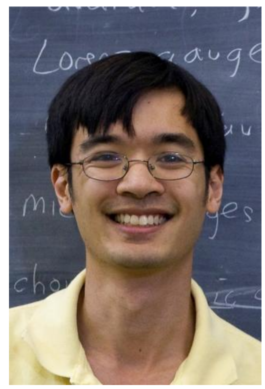
Bryan Birch

Peter Swinneton-Dyer

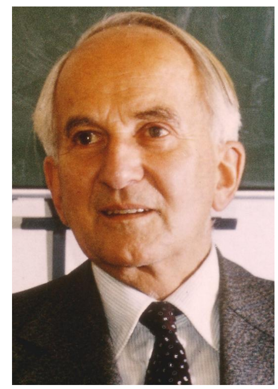
[2] Andrew Wiles from FLT.

After this preliminary version, Wight moves on to a more detailed version of BSD, trying to clarify and relate '*what is the structure of the infinite set?*' and '*how zero is zero?*'. Therefore he defines the rank of E (the number of solutions required to generate all the rational solutions) and the order of a zero. The BSD then says that these are the same: the order of the zero at $s = 1$ for $L_E(s)$ equals the rank of E. The Coates-Wiles theorem was superseded in 2015 by a paper of Bhargava and Shankar who proved that a considerable part of elliptic curves have rank 0 and therefore satisfy BSD.

One of the Erdős conjectures is about arithmetic progression. If the sum of the inverses of the numbers in a subset A of positive integers diverges to infinity, then A contains an arithmetic sequence of any length. When A is a set of primes (the sum of reciprocals diverges), then the Green-Tao theorem (2004) says that, no matter how large you choose n, you will always find a sequence of n successive equally spaced primes. Terence Tao received the Fields Medal in 2006. The conjecture thus says that such statement should hold for any set of positive integers, not only primes. Erdős offered in 1976 a prize of 5000 dollar for a proof of his conjecture although he never cared about where to find the money when he awarded such prizes, but the amount somehow reflected a level of importance of the problem.



Terrence Tao



Lothar Collatz

To conclude, the book lists problem "*easy to understand but impossible to solve*". Erdős once said "Children can ask questions about primes which grown men cannot answer". So there are some more problems that are less in the focus of mathematicians, mostly because nobody has a clue on how to tackle them. There is the Collatz conjecture ("*1930's version of angry birds*"). "*Back in the 1920's and 30's, the world was populated by savages who hadn't yet discovered the massive societal value of devoting hundreds of hours to noble endeavors like Angry Birds or Addiction Solitaire. To waste time [...] they had to find a simple mathematical problem that was as addictive as it was impossible*". Collatz's algorithm goes as follows. Pick a number x (positive integer), if it is even, divide by 2 and if odd, replace it by $3x+1$ and repeat. The claim is that this will always arrive at 1 and thus end with the cycle 1, 4, 2. It is an illustration of nonlinear dynamics create by a simple algorithm producing quite unpredictable behavior. It is the number theoretic version of a chaotic dynamical system.

Goldbach's conjecture appears in a 1742 letter that Christian Goldbach wrote to Euler: Every even integer > 2 can be written as the sum of two primes. He also had a weaker ternary version: Every integer > 7 can be written as the sum of 3 primes. But that is trivial, since subtracting 3 gives an even number that can be written as the sum of two primes by the even version. So it remains to prove the original one. The proof of the weak version was however given independently for all odd numbers larger than an impossible large number. In 2013, this bound was reduced to 10^{30} and the finitely many remaining cases could be treated by a computer. QED.

Christian Goldbach

The twin prime conjecture is about the existence of infinitely many prime numbers that differ by 2, a question already raised by Euclid. No progress was made for 2000 years. Then in 1849 de Polignac generalized the problem for pairs of successive primes that differ by some k. These got names like 'twins' (2), 'cousins' (4), 'co-workers' (8). While for 16, Wright calls them '*two people that saw each other on the street but haven't really talked to each other but wouldn't oppose to it*'.

Perfect numbers are numbers that are equal to the sum of their proper divisors. These numbers are rather rare. A list of 49 is known in June 2016. It starts with 6 and the 49th has 44,677,235 digits but it is conjectured there are infinitely many perfect numbers. There is a relation with Mersenne primes, i.e., primes of the form $2^n - 1$ for particular integers of n. It is known that if $2^n - 1$ is a Mersenne prime, then $2^{n-1}(2^n - 1)$ is a perfect number. Only, it is not known that there are infinitely many Mersenne primes. Neither is it known if there exists an odd perfect number.

Adhemar Bultheel